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HYPERBOLIC STRUCTURE PRESERVING ISOMORPHISMS OF MARKOV SHIFTS. II

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ABSTRACT

This paper is a continuation of [14] and deals with metric isomorphisms of Markov shifts which are finitary and hyperbolic structure preserving. We prove that the β-function introduced by S. Tuncel in [15] is an invariant of such isomorphisms. Following [5] this result is extended to Gibbs measures arising from functions with summable variation. Finally we prove that, for any C^2 Axiom A diffeomorphism on a basic set Ω, and for any equilibrium state associated with a Hölder continuous function on Ω, the Markov shifts arising from different Markov partitions of Ω are isomorphic via a finitary, hyperbolic structure preserving isomorphism. This fact leads to a rich class of examples of such isomorphisms (other examples are provided by finitary isomorphisms of Markov shifts with finite expected code lengths — cf. [14]).

1. Introduction

This paper continues the investigation in [14] of measure preserving isomorphisms of Markov shifts which respect the hyperbolic structure of the shift spaces almost everywhere. As was shown in [14], a hyperbolic structure preserving isomorphism $\varphi : X_P \to X_Q$ of two Markov shifts T_P and T_Q on shift spaces (X_P, m_P) and (X_Q, m_Q), respectively, need not be finitary, and finitary isomorphisms of T_P and T_Q need not preserve their hyperbolic structures. However, there exists a class of finitary isomorphisms which preserve the hyperbolic structures of the shift spaces, the so-called isomorphisms with finite expected code lengths. Finitary isomorphisms with finite expected code lengths were studied by W. Parry, W. Krieger and the author in several papers (cf. [7], [8], [9], [13]), in which various obstructions to the existence of such isomorphisms were obtained. All the invariants providing these obstructions turned out to be invariants of a more general class of finitary isomorphisms which is characterized

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by the condition that certain functions associated with the isomorphism (the functions a^*_φ, m^*_φ, $a^*_{\varphi^{-1}}$ and $m^*_{\varphi^{-1}}$ described in Section 2 of this paper) should be finite a.e., but there did not seem to be a natural interpretation of this finiteness condition in terms of the topological or coding properties of the isomorphism. Some of these invariants (the groups Γ_P and Δ_P, the distinguished generator of the quotient group Γ_P/Δ_P, and the information cocycle) were shown in [14] to be invariants of the hyperbolic structure of the Markov shift and to have nothing to do with any finitariness assumptions, but others (in particular the β-function of the Markov shift introduced by S. Tuncel in [15]) remain tied to this finiteness condition.

In this paper we prove that, for a finitary isomorphism $\varphi : X_P \to X_Q$ of two ergodic Markov shifts T_P and T_Q, the finiteness of these functions a^*_φ, m^*_φ, $a^*_{\varphi^{-1}}$ and $m^*_{\varphi^{-1}}$ is equivalent to φ being hyperbolic structure preserving. Hence the β-function of a Markov shift is an invariant of finitary, hyperbolic structure preserving isomorphisms. In particular, distinct Bernoulli shifts cannot be isomorphic in this manner (cf. [13]). The methods used in this paper yield these results in "asymmetric" form, thus allowing conclusions about the particular direction in which a metric isomorphism of Markov shifts must fail to be both finitary and hyperbolic structure preserving. Earlier asymmetric results concerning Δ_P, Γ_P and the information cocycle can be found in [7] and [14]. Here we show that, if there exists a metric isomorphism $\varphi : X_P \to X_Q$ of two Markov shifts T_P and T_Q which is continuous when restricted to a set of full measure and which sends (up to a null set) the stable and unstable manifolds of every point x in X_P into the stable and unstable manifolds of $\varphi(x)$ in X_Q then the β-functions β_P and β_Q of the two shifts satisfy that $\beta_P(t) \leqq \beta_Q(t)$ for every $t \in \mathbf{R}$ (Theorem 5.1). This result about the β-function is shown to hold more generally for finitary hyperbolic structure homomorphisms which preserve Gibbs measures arising from functions with summable variation (Theorem 6.2). An earlier result in this direction was obtained by A. Harding in [5], where it was shown that the β-function is invariant for finitary isomorphisms of such Gibbs measures with finite expected code lengths. The proof of Theorem 6.2 is quite similar to that of Theorem 5.1, but technically more involved. In the interest of conceptual clarity we present the proofs for Markov measures in full in sections 4 and 5 and restrict ourselves to indicating the changes necessary when dealing with Gibbs measures in section 6.

The last section also shows how finitary, hyperbolic structure preserving isomorphisms of Markov shifts equipped with Gibbs measures arise naturally in a geometric context: consider an Axiom A diffeomorphism, restricted to a basic set Ω, with an equilibrium state arising from a Hölder continuous function $\chi:\Omega\to\mathbf{R}$. Then any two Markov shifts associated with different Markov partitions of Ω are isomorphic in this manner. Furthermore, consider two Axiom A diffeomorphisms f_i, $i=1,2$, restricted to basic sets Ω_i, with equilibrium states arising from Hölder continuous functions on Ω_i. If these diffeomorphisms have Markov partitions which give rise to Markov shifts which are isomorphic via a finitary, hyperbolic structure preserving map, then there exist null sets $N_i\subset\Omega_i$ and a measure preserving homeomorphism $\psi:\Omega_1\setminus N_1\to\Omega_2\setminus N_2$ with $\psi\cdot f_1=f_2\cdot\psi$ on $\Omega_1\setminus N_1$ and such that, for every $x\in\Omega_1$,

$$\psi(W^s_\omega\setminus N_1)=W^s_{\psi(\omega)}\setminus N_2\quad\text{and}\quad\psi(W^u_\omega\setminus N_1)=W^u_{\psi(\omega)}\setminus N_2,$$

where W^s_ω and W^u_ω are the stable and unstable manifolds in Ω_i of a point $\omega\in\Omega_i$. For a hyperbolic, linear automorphism A of the n-torus $\mathbf{R}^n/\mathbf{Z}^n$ one obtains the existence of a null set $N\subset\mathbf{R}^n/\mathbf{Z}^n$ and of a homeomorphism $\psi:\mathbf{R}^n/\mathbf{Z}^n\setminus N\to\mathbf{R}^n/\mathbf{Z}^n\setminus N$ such that $\psi\cdot A=A^{-1}\cdot\psi$ on $\mathbf{R}^n/\mathbf{Z}^n\setminus N$ and

$$\psi(W^s_\omega\setminus N)=W^u_{\psi(\omega)}\setminus N,\quad\psi(W^u_\omega\setminus N)=W^s_{\psi(\omega)}\setminus N\qquad\text{for every }\omega\in\mathbf{R}^n/\mathbf{Z}^n.$$

Note that W^s_ω and W^u_ω may have different dimensions.

2. Finitary isomorphisms of Markov shifts

Let $P=(P(i,i'),1\leqq i,i'\leqq k)$, $k\geqq 2$, be an irreducible, stochastic matrix and let $p^-=(p^-(i),1\leqq i\leqq k)$ be its left eigenvector with eigenvalue 1 and $\Sigma_{1\leqq i\leqq k}p^-(i)=1$. We write X_P for the space of sequences $x=(x_n,n\in\mathbf{Z})\in\{1,\ldots,k\}^{\mathbf{Z}}$ with $P(x_n,x_{n+1})>0$ for all $n\in\mathbf{Z}$, denote by $T_p:X_P\to X_P$ the shift $(T_Px)_n=x_{n+1}$, and define a T_P-invariant Markov probability measure m_P on X_P by setting

$$m_P(C)=p^-(i_0)P(i_0,i_1)\cdots P(i_{s-1},i_s)\tag{2.1}$$

for every cylinder set $C=[i_0,\ldots,i_s]_r=\{x\in X_P:x_{r+m}=i_m\text{ for }0\leqq m\leqq s\}$. The automorphism T_P of the probability space (X_P,m_P) is ergodic and is called the *Markov shift* associated with the stochastic matrix P. If $Q=(Q(j,j'),1\leqq j,j'\leqq l)$ is another irreducible, stochastic matrix and T_Q the associated Markov shift on the shift space (X_Q,m_Q), the shifts T_P and T_Q are said to be *isomorphic* if there exists a measure preserving isomorphism $\varphi:X_P\to X_Q$ with

$$\varphi T_P=T_Q\varphi\qquad m_P\text{-a.e.}\tag{2.2}$$

An isomorphism $\varphi:X_P\to X_Q$ satisfying (2.2) is called *almost continuous* if there exists a null set $N_P \subset X_P$ such that the restriction of φ to $X_P \setminus E_P$ is continuous, and *finitary* if both φ and φ^{-1} are almost continuous. An isomorphism φ is almost continuous if and only if there exists a shift-invariant null set $N_P \subset X_P$ and functions $a_\varphi, m_\varphi : X_P \setminus N_P \to \mathbf{N}$ with the following properties.

(2.3) The restriction of φ to $X_P \setminus N_P$ is continuous and injective,

$$\varphi(T_P x) = T_Q \varphi(x) \qquad \text{for every } x \in X_P \setminus N_P, \tag{2.4}$$

$$\begin{gathered}\varphi(x)_0 = \varphi(x')_0 \qquad \text{for all } x, x' \in X_P \setminus N_P \\ \text{with } x_i = x'_i \quad \text{for } -m_\varphi(x) \leqq i \leqq a_\varphi(x),\end{gathered} \tag{2.5}$$

and

(2.6) the functions a_φ, m_φ are "minimal" in the sense that, whenever $a'_\varphi, m'_\varphi : X_P \setminus N_P \to \mathbf{N}$ satisfy (2.5), then $a'_\varphi(x) \geqq a_\varphi(x)$ and $m'_\varphi(x) \geqq m_\varphi(x)$ for all $x \in X_P \setminus N_P$.

Note that the functions a_φ and m_φ must be continuous on $X_P \setminus N_P$. Put, for every $x \in X_P \setminus N_P$,

$$a^*_\varphi(x) = \sup_{n \geqq 0} (a_\varphi(T_P^{-n} x) - n) \tag{2.7}$$

and

$$m^*_\varphi(x) = \sup_{n \geqq 0} (m_\varphi(T_P^{n} x) - n) \tag{2.8}$$

(cf. [7], [9] and [13]). Since

$$a^*_\varphi(T_P^{-1} x) = \sup_{n \geqq 1} (a_\varphi(T_P^{-n} x) - n) + 1 \leqq a^*_\varphi(x) + 1 \tag{2.9}$$

and

$$m^*_\varphi(T_P x) = \sup_{n \geqq 1} (m_\varphi(T_P^{n} x) - n) + 1 \leqq m^*_\varphi(x) + 1 \tag{2.10}$$

for every $x \in X_P \setminus N_P$, the sets

$$A_P = \{x \in X_P \setminus N_P : a^*_\varphi(x) < \infty\} \tag{2.11}$$

and

$$M_P = \{x \in X_P \setminus N_P : m^*_\varphi(x) < \infty\} \tag{2.12}$$

are T_P-invariant. For every $x, x' \in A_P$ with $x_i = x'_i$ for $i \leqq a^*_\varphi(x)$, (2.13) $$\varphi(x)_j = \varphi(x')_j \qquad \text{for all } j \leqq 0.$$

Similarly, if x, $x' \in M_P$ and $x_i = x'_i$ for $i \geqq -m^*_\varphi(x)$, then

(2.14) $$\varphi(x)_j = \varphi(x)_j \qquad \text{for all } j \geqq 0.$$

If φ is finitary we denote by N_Q, $a_{\varphi^{-1}}$, $m_{\varphi^{-1}}: X_Q \setminus N_Q \to \mathbf{N}$, $a^*_{\varphi^{-1}}$ and $m^*_{\varphi^{-1}}$ the corresponding objects for φ^{-1}, and we may assume that φ is a homeomorphism of $X_P \setminus N_P$ onto $X_Q \setminus N_Q$.

3. Hyperbolic structure preserving isomorphisms

Let $P = (P(i,i'), 1 \leqq i, i' \leqq k)$ be an irreducible, stochastic matrix and let T_P be the Markov shift on the probability space (X_P, m_P). For every $x \in X_P$ and $n \in \mathbf{Z}$ we put

(3.1) $$W^s_x(P,n) = \{x' \in X_P : x'_k = x_k \text{ for all } k \geqq n\}$$

and

(3.2) $$W^u_x(P,n) = \{x' \in X_P : x'_k = x_k \text{ for all } k \leqq n\}.$$

The *stable* and *unstable manifolds* W^s_x and W^u_x of x are given by

(3.3) $$W^s_x = \bigcup_{n \in \mathbf{Z}} W^s_x(P,n)$$

and

(3.4) $$W^u_x = \bigcup_{n \in \mathbf{Z}} W^u_x(P,n).$$

If $(Q(j,j'), 1 \leqq j, j' \leqq l)$ is another irreducible, stochastic matrix, T_Q the corresponding Markov shift on (X_Q, m_Q), and $\varphi : X_P \to X_Q$ a measure preserving isomorphism satisfying (2.2) then φ is called a *hyperbolic structure homomorphism* if there exists a null set $E_P \subset X_P$ such that

(3.5) $$\varphi(W^u_x \setminus E_P) \subset W^u_{\varphi(x)}$$

and

(3.6) $$\varphi(W^s_x \setminus E_P) \subset W^s_{\varphi(x)}$$

for every $x \in X_P$. If null sets $E_P \subset X_P$ and $E_Q \subset X_Q$ can be chosen so that

(3.7) $$\varphi(W^u_x \setminus E_P) = W^u_{\varphi(x)} \setminus E_Q$$

and

$$\varphi(W_x^s \setminus E_P) = W_{\varphi(x)}^s \setminus E_Q \tag{3.8}$$

then φ is said to be *hyperbolic structure preserving* — cf. [14]. Following [14] we define equivalence relations R_P, $R'_P \subset X_P \times X_P$ by setting, for every $x \in X_P$,

$$R_P(x) = \bigcup_{k,l \in \mathbf{Z}} (W_{T_P^k x}^u \cap W_{T_P^l x}^s) \tag{3.9}$$

and

$$R'_P(x) = W_x^u \cap W_x^s, \tag{3.10}$$

where $R_P(x)$ and $R'_P(x)$ are the equivalence classes of a point $x \in X_P$ in the relation R_P and R'_P, respectively. These equivalence relations are nonsingular, and the Radon–Nikodym derivative $r_P : R_P \to \mathbf{R}_+$ of R_P is given by

$$r_P(x, x') = dm_P(x)/dm_P(x') = \prod_{-m \leqq s \leqq n-1} P(x_s, x_{s+1}) \Big/ \prod_{-m' \leqq s \leqq n'-1} P(x'_s, x'_{s+1}) \tag{3.11}$$

for every $(x, x') \in R_P$, where m, m', n, n' are natural numbers satisfying

$$x_{-m-s} = x'_{-m'-s} \quad \text{and} \quad x_{n+s} = x'_{n'+s} \tag{3.12}$$

for all $s \geqq 0$ (cf. [14], (2.6) and (2.9)). From (3.5), (3.6), (3.9) and (3.10) it is clear that

$$\varphi(R_P(x)) \subset R_Q(x)) \quad \text{and} \quad \varphi(R'_P(x)) \subset R'_Q(\varphi(x)) \tag{3.13}$$

for m_P-a.e. $x \in X_P$, where the subscript Q indicates the corresponding objects on X_Q. We write G_P (resp. F_P) for the group of all nonsingular automorphisms V of (X_P, m_P) which satisfy that $Vx \in R_P(x)$ (resp. $Vx \in R'_P(x)$) for m_P-a.e. $x \in X_P$, and denote by F_P^0 the subgroup of measure preserving elements in F_P. Formula (3.13) shows that

$$\varphi G_P \varphi^{-1} \subset G_Q, \quad \varphi F_P \varphi^{-1} \subset F_Q \quad \text{and} \quad \varphi F_P^0 \varphi^{-1} \subset F_Q^0. \tag{3.14}$$

4. Finitary isomorphisms which preserve the hyperbolic structure

4.1. THEOREM. *Let P and Q be irreducible, stochastic matrices and let $\varphi : X_P \to X_Q$ be an almost continuous isomorphism of the associated Markov shifts T_P and T_Q. Choose a null set $N_P \subset X_P$ and functions a_φ, $m_\varphi : X_P \setminus N_P \to \mathbf{N}$ satisfying (2.3)–(2.6) and define a_φ^*, $m_\varphi^* : X_P \setminus N_P \to \mathbf{N} \cup \{\infty\}$ by (2.7) and (2.8). The following conditions are equivalent:*

(1) *φ is a hyperbolic structure homomorphism,*

(2) *the functions a_φ^*, m_φ^* are finite m_P-a.e. on X_P.*

PROOF. First assume that φ is almost continuous and that a_φ^* and m_φ^* are finite a.e. The proof of Proposition 4.4 in [14] shows that φ is a hyperbolic structure homomorphism. In order to prove the converse we assume that $\varphi : X_P \to X_Q$ is almost continuous and a homomorphism of the hyperbolic structures of X_P and X_Q. Suppose, for example, that $a_\varphi^* = \infty$ on a set of positive measure and hence, by Section 2, m_P-a.e. Let $\alpha_P = \{[1]_0, \ldots, [k]_0\}$ be the state partition of X_P and let, for every $n \in \mathbf{Z}$, $\mathscr{A}_P(n)$ be the σ-algebra generated by $\{T_P^{-m}\alpha_P : m \leqq n\}$. We write $\{\nu_x^{(P,n)} : x \in X_P\}$ for the decompositions of the measure m_P with respect to the σ-algebra $\mathscr{A}_P(n)$ and define the corresponding objects α_Q, $\mathscr{A}_Q(n)$ and $\{\nu_y^{(Q,n)} : y \in X_Q\}$ for the space X_Q. Since φ is a hyperbolic structure homomorphism we know that, for m_P-a.e. $x \in X_P$, and for every $n \in \mathbf{Z}$, $\nu_x^{(P,n)}(\bigcup_{m \in \mathbf{Z}} \varphi^{-1}(W^u_{\varphi(x)}(Q, m))) = 1$. Hence we can find, for every $\varepsilon > 0$, an integer $M = M(\varepsilon) \in \mathbf{N}$ such that, for every $n \in \mathbf{Z}$,

$$m_P(D^u(n, \varepsilon)) > 1 - \varepsilon, \tag{4.1}$$

where

$$D^u(n, \varepsilon) = \{x \in X_P : \nu_x^{(P,n)}(\varphi^{-1}(W^u_{\varphi(x)}(Q, n - M))) > 1 - \varepsilon\} \tag{4.2}$$

(cf. [14], (3.16) and (3.18)). For every $n \in \mathbf{Z}$ we put

$$F_0^+(n) = \{V \in F_P^0 : (Vx_i) = x_i \ m_P\text{-a.e., for every } i \leqq n\}. \tag{4.3}$$

Let $\varepsilon = \frac{1}{4}$, choose $M(\frac{1}{4})$ according to (4.1) and (4.2), and put $n = M(\frac{1}{4})$. Suppose that we can find an automorphism $V \in F_0^+(M(\frac{1}{4}))$ with the property that

$$M_P(\{x \in X_P : \varphi(Vx)_j \neq \varphi(x)_j \text{ for some } j \leqq 0\}) = 1. \tag{4.4}$$

Since $V \in F_0^+(M(\frac{1}{4}))$, V preserves $\nu_x^{(P,M(1/4))}$ for m_P-a.e. $x \in X$, and equation (4.4) implies that

$$\begin{aligned} &\nu_x^{(P,M(1/4))}(\{x' \in X_P : \varphi(x') \in W^u_{\varphi(x)}(Q, 0)\}) \\ &\quad = \nu_x^{(P,M(1/4))}(\{x' \in W^u_x(P, M(\tfrac{1}{4})) : \varphi(Vx') \in W^u_{\varphi(x)}(Q, 0)\}) < \tfrac{1}{4} \end{aligned}$$

for m_P-a.e. $x \in D^u(M(\frac{1}{4}), \frac{1}{4})$. This violates our choice of $M(\frac{1}{4})$ and $n = M(\frac{1}{4})$ in (4.1) and (4.2), and the resulting contradiction shows that $a_\varphi^* < \infty$ m_P-a.e.

The automorphism V is constructed by an exhaustion argument. Suppose that we have defined V on a subset $E \subset X_P$, and that V has the following properties there:

$$\begin{aligned} &V \text{ is an } m_P\text{-preserving Borel automorphism of } E, \\ &\quad \text{and } V^2x = x \text{ for every } x \in E, \end{aligned} \tag{4.5}$$

(4.6) $Vx \in R'_P(x)$ for every $x \in E$,

(4.7) $(Vx)_i = x_i$ for every $x \in E$ and $i \leqq M(\frac{1}{4})$,

and

(4.8) for every $x \in E$, $\varphi(Vx)_j \neq \varphi(x)_j$ for some $j \leqq 0$.

Since $a^*_\varphi = \infty$ on $X_P \setminus N$ we can find, for m_P-a.e. $x \in X_P$, and for every $m > 0$, a point $x' \in X_P$ such that $x_i = x'_i$ for all $i \leqq m$, but $\varphi(x)_j \neq \varphi(x')_j$ for some $j \leqq 0$. An elementary argument allows us to choose, for every $m > 0$, an automorphism $W_m \in F^+_0(P, M(\frac{1}{4}) + m)$ such that $W^2_m = 1$ (the identity map on X_P) and there exists, for m_P-a.e. $x \in X_P$, and integer $j \leqq 0$ with $\varphi(W_m x)_j \neq \varphi(x)_j$. Assume that $F = X_P \setminus E$ has positive measure. Since $\lim_m m_P(W_m F \triangle F) = 0$ and hence $\lim_m m_P(W_m F \cap F) = m_P(F) > 0$, we can choose an integer $m' > 0$ and a Borel set B such that $m_P(B) > 0$, $W_m B \cap B = \emptyset$ and $W_{m'} B \cup B \subset F$. Let $E' = E \cup W_{m'} B$ and define a Borel automorphism $V' : E' \to E'$ by setting $V'x = Vx$ for $x \in E$ and $V'x = W_{m'} x$ for $x \in B \cup W_{m'} B$. The automorphism V' satisfies (4.5)–(4.8) with E' replacing E. A countable transfinite induction argument shows that there exists a Borel set $E \subset X_P$ with $m_P(E) = 1$ and a Borel automorphism V of E satisfying (4.5)–(4.8). As we have pointed out earlier this implies that $a^*_\varphi(x) < \infty$ for m_P-a.e. $x \in X$. Similarly one proves that $m^*_\varphi < \infty$ m_P-a.e. □

4.2. Corollary. *Let P and Q be irreducible, stochastic matrices and let $\varphi : X_P \to X_Q$ be an isomorphism of the Markov shifts T_P and T_Q. The map φ is an almost continuous hyperbolic structure homomorphism if and only if there exists a null set $N_P \subset X_P$ and functions a^*_φ, $m^*_\varphi : X_P \setminus N_P \to \mathbf{N}$ such that, for all x, $x' \in X_P \setminus N_P$,*

(4.9) $\varphi(x)_j = \varphi(x')_j$ *for all* $j \leqq 0$ *whenever* $x_i = x'_i$ *for all* $i \leqq a^*_\varphi(x)$

and

(4.10) $\varphi(x)_j = \varphi(x')_j$ *for all* $j \geqq 0$ *whenever* $x_i = x'_i$ *for all* $i \geqq -m^*_\varphi(x)$.

4.3. Corollary. *Let P and Q be irreducible, stochastic matrices and let $\varphi : X_P \to X_Q$ be a finitary isomorphism of the Markov shifts T_P and T_Q. Define a^*_φ, $m^*_\varphi : X_P \setminus N_P \to \mathbf{N} \cup \{\infty\}$ and, by analogy, $a^*_{\varphi^{-1}}$, $m^*_{\varphi^{-1}} : X_P \setminus N_Q \to \mathbf{N} \cup \{\infty\}$ by* (2.3)–(2.8). *The following conditions are equivalent*:

(1) *φ is hyperbolic structure preserving,*

(2) *the functions a^*_φ, m^*_φ and $a^*_{\varphi^{-1}}$, $m^*_{\varphi^{-1}}$ are finite m_P-a.e. on X_P and X_Q, respectively.*

5. The β-function and other invariants

Let $P = (P(i,i'), 1 \leqq i, i' \leqq k)$, $k \geqq 2$, be an irreducible, stochastic matrix. For every $t \in \mathbf{R}$ we write $P^{(t)} = (P(i,i')^t, 1 \leqq i, i' \leqq k)$ for the matrix obtained by raising every nonzero entry of P to the power t. Following [15] we denote by $\beta_P(t)$ the maximal eigenvalue of $P^{(t)}$. The function $\beta_P : \mathbf{R} \to \mathbf{R}$ is analytic (cf. [10]) and plays an important role in certain coding problems of Markov shifts (cf. [10], [13] and [15]). We shall need the following characterization of β_P, which is an immediate consequence of [10]: for every $x \in X_P$, $t \in \mathbf{R}$,

$$\beta_P(t) = \lim_n \left\{ \sum_{x' \in W^u_x(P,0) \cap W^s_x(P,n)} P(x'_0, x'_1)^t \cdots P(x'_{n-1}, x'_n)^t \right\}^{1/n}, \tag{5.1}$$

and the limit in (5.1) is uniform in x.

5.1. THEOREM. *Let P and Q be irreducible, stochastic matrices and let T_P and T_Q be the Markov shifts on (X_P, m_P) and (X_Q, m_Q), respectively. Assume that there exists an almost continuous isomorphism $\varphi : X_P \to X_Q$ of T_P and T_Q which is a hyperbolic structure homomorphism. Then $\beta_P(t) \leqq \beta_Q(t)$ for every $t \in \mathbf{R}$.*

PROOF. According to Theorem 4.1 the functions a^*_φ and m^*_φ are finite m_P-a.e. As in [9] and [13] we choose an integer $M > 0$ and a cylinder set $C = [i_0, \ldots, i_{2M}]_{-M} \subset X_P$ with the property that $D = C \cap \{x \in X_P : a^*_\varphi(x) \leqq M, m^*_\varphi(x) \leqq M\}$ has positive measure. For every $n \geqq 0$ we set

$$D(n) = \bigcup_{x \in D \cap T_P^{-n} D} W^u_x(P, M) \cap W^s_x(P, n - M) \tag{5.2}$$

and define a map $V_n : D(n) \to X_P$ by setting

$$(V_n x)_i = x_i \quad \text{for } i \leqq 0 \quad \text{and} \quad (V_n x)_i = x_{i+n} \quad \text{for } i \geqq 0. \tag{5.3}$$

Note that, for every $x \in D(n)$, $(V_n x)_i = x_i$ for all $i \leqq a^*_\varphi(x) \leqq M$ and $(V_n x)_i = (T_P^n x)_i$ for all $i \geqq -m^*_\varphi(T_P^n x) \geqq -M$. From (2.13) and (2.14) it is clear that there exists a null set $N \subset X_P$ such that, for every $n \geqq 0$ and every $x \in D(n) \setminus N$,

$$\varphi(V_n x)_j = \varphi(x)_j \qquad \text{for all } j \leqq 0, \tag{5.4}$$

$$\varphi(V_n x)_j = \varphi(x)_{j+n} \qquad \text{for all } j \geqq 0, \tag{5.5}$$

and

$$\varphi(W^u_x(P, M) \cap W^s_x(P, n - M)) \subset W^u_{\varphi(x)}(Q, 0) \cap W^s_{\varphi(x)}(Q, n). \tag{5.6}$$

Since φ is measure preserving, and since $(x, V_n x) \in R_P$ for every $x \in D(n) \setminus N$, (5.4) and (5.5) allow us to assume that

$$
\begin{aligned}
dm_P(x)/dm_P(V_nx) &= P(x_0,x_1)\cdots P(x_{n-1},x_n) \\
&= dm_Q(\varphi(x))/dm_Q(\varphi(V_nx)) \qquad (5.7)\\
&= Q(\varphi(x)_0,\varphi(x)_1)\cdots Q(\varphi(x)_{n-1},\varphi(x)_n)
\end{aligned}
$$

for every $n \geqq 1$ and every $x \in D(n)\setminus N$ (cf. (3.11) and (3.12)). Hence we have, for every $t \in \mathbf{R}$, $n \geqq 1$ and $x \in D(n)\setminus N$,

$$
\begin{aligned}
&\left\{\sum_{x' \in W_x^u(P,M)\cap W_x^s(P,n-M)} P(x'_0,x'_1)^t\cdots P(x'_{n-1},x'_n)^t\right\}^{1/n} \qquad (5.8)\\
&\quad\leqq \left\{\sum_{y\in W^u_{\varphi(x)}(Q,0)\cap W^s_{\varphi(x)}(Q,n)} Q(y_0,y_1)^t\cdots Q(y_{n-1},y_n)^t\right\}^{1/n}.
\end{aligned}
$$

An easy calculation shows that, for every $\varepsilon > 0$, there exists an integer $n' > 0$ with

$$
\begin{aligned}
&\Bigg|\left\{\sum_{x' \in W_x^u(P,M)\cap W_x^s(P,n-M)} P(x'_0,x'_1)^t\cdots P(x'_{n-1},x'_n)^t\right\}^{1/n} \qquad (5.9)\\
&\quad - \left\{\sum_{x' \in W_x^u(P,0)\cap W_x^s(P,n)} P(x'_0,x'_1)^t\cdots P(x'_{n-1},x'_n)^t\right\}^{1/n}\Bigg| < \varepsilon
\end{aligned}
$$

for every $n \geqq n'$ and every $x \in D(n)$. By comparing (5.8) and (5.9) with (5.1) we conclude that $\beta_P(t) \leqq \beta_Q(t)$, as claimed. □

5.2. Corollary [13]. *Let P and Q be irreducible, stochastic matrices and let T_P and T_Q be the corresponding Markov shifts on (X_P, m_P) and (X_Q, m_Q), respectively. Assume that there exists a finitary, hyperbolic structure preserving isomorphism $\varphi : X_P \to X_Q$ of T_P and T_Q. Then $\beta_P(t) = \beta_Q(t)$ for every $t \in \mathbf{R}$.*

5.3. Corollary [13]. *Let $p = (p(1),\dots,p(k))$ and $q = (q(1),\dots,q(1))$ be probability vectors with all entries nonzero and let T_P and T_Q be the Bernoulli shifts based on p and q, respectively. Assume that there exists a finitary, hyperbolic structure preserving isomorphism φ of T_P and T_Q. Then $k = l$ and q is obtained by permuting the entries of p.*

Let $P = (P(i,i'), 1 \leqq i, i' \leqq k)$ be an irreducible, stochastic matrix and let Γ_P be the multiplicative group of positive real numbers consisting of all ratios of the form

$$\prod_{0\leqq s\leqq m} P(i_s,i_{s+1}) \Big/ \prod_{0\leqq s\leqq n} P(i'_s,i'_{s+1}) \tag{5.10}$$

with $m, n > 0$, $1 \leqq i_s, i'_s \leqq k$, $i_0 = i_{m+1} = i'_0 = i'_{n+1}$, and such that both the numerator and the denominator in (5.10) are nonzero. The subgroup of Γ_P consisting of all ratios of the form (5.10) with $m = n$ will be denoted by Δ_P (cf. [7] and [9]). We also recall from [9] that the quotient group Γ_P/Δ_P is cyclic: there exists a positive real number c_P with

$$\prod_{0 \leqq s < m} P(i_s, i_{s+1}) \in c_P^m \Delta_P \tag{5.11}$$

whenever $m > 0$, $1 \leqq i_s \leqq k$, $i_0 = i_m$, and $P(i_0, i_1) \cdots P(i_{m-1}, i_m) > 0$. Now let Q be a second irreducible, stochastic matrix, denote by Γ_Q, Δ_Q and c_Q the corresponding objects for Q, and assume that there exists an almost continuous isomorphism $\varphi : X_P \to X_Q$ of the Markov shifts T_P and T_Q which is a hyperbolic structure homomorphism. Remark 2.2 in [14] shows that $\Gamma_P \subset \Gamma_Q$, $\Delta_P \subset \Delta_Q$ and $c_P^d \Delta_P \subset c_Q^d \Delta_Q$, where d denotes the period of P (and Q), and from Theorem 4.1 in this paper we know that $\beta_P(t) \leqq \beta_Q(t)$ for every $t \in \mathbf{R}$. We conclude this section with some examples of pairs of Markov shifts T_P and T_Q which do not admit any (almost continuous) hyperbolic structure homomorphisms (and, in particular, no finitary isomorphisms with finite expected code length) $\varphi : X_P \to X_Q$.

5.4. Example (Meshalkin). Let T_P and T_Q be Bernoulli shifts based on the probability vectors $p = (\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ and $q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, respectively. Then $\Gamma_P = \{2^n : n \in \mathbf{Z}\}$, $\Gamma_Q = \{4^n : n \in \mathbf{Z}\}$, so there cannot exist a hyperbolic structure homomorphism $\varphi : X_P \to X_Q$ (cf. [7]).

5.5. Example. Let $p = (\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, 12 * \frac{1}{64})$ (the notation $12 * \frac{1}{64}$ indicates 12 entries all of which are equal to $\frac{1}{64}$), $q = (\frac{1}{2}, \frac{1}{4}, 6 * \frac{1}{32}, 8 * \frac{1}{128})$, and let T_P and T_Q be the associated Bernoulli shifts. Then $\Gamma_P = \Gamma_Q = \Delta_P = \Delta_Q = \{2^n : n \in \mathbf{Z}\}$, and $\beta_P(t) - \beta_Q(t) = 16^{-t} \cdot (1 - 2 \cdot 2^{-t})^3$. Since $\beta_P(0) < \beta_Q(0)$ and $\beta_P(2) > \beta_Q(2)$, there exist no isomorphisms $\varphi : X_P \to X_Q$ of these shifts which are almost continuous hyperbolic structure homomorphisms in either direction.

5.6. Example ([3]). Let $p = (\frac{1}{4}, \frac{1}{8}, 2 * \frac{1}{16}, \frac{1}{32}, 25 * \frac{1}{64}, \frac{1}{128}, 18 * \frac{1}{256})$, $q = (\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 9 * \frac{1}{32}, \frac{1}{64}, 33 * \frac{1}{128}, 2 * \frac{1}{256})$, and let T_P and T_Q be the Bernoulli shifts based on these vectors. Then $\Gamma_P = \Gamma_Q = \Delta_P = \Delta_Q = \{2^n : n \in \mathbf{Z}\}$ and $\beta_P(t) - \beta_Q(t) = 16^{-t} \cdot (1 - 2 \cdot 2^{-t})^4 \geqq 0$ for all $t \in \mathbf{R}$, and there cannot exist an isomorphism $\varphi : X_P \to X_Q$ of T_P and T_Q such that φ is an almost continuous hyperbolic structure homomorphism.

In all these examples there exists a finitary isomorphism $\varphi : X_P \to X_Q$ of the Bernoulli shifts T_P and T_Q, since the shifts have equal entropy (cf. [6]).

6. Gibbs measures and Axiom A diffeomorphisms

Let $P = (P(i,i'), 1 \leqq i, i' \leqq k)$ be an irreducible 0–1 matrix and let the corresponding shift space X_P, the shift T_P, and the equivalence relations R_P and R'_P be given exactly as in the case of a stochastic matrix P. For every continuous function $g : X_P \to \mathbf{R}$ and every $n \geqq 0$ we put

$$\operatorname{var}_n(g) = \sup\{|g(x) - g(x')| : x, x' \in X_P \text{ and } x_i = x'_i \text{ for all } |i| \leqq n\}.$$

A continuous $g : X_P \to \mathbf{R}$ is said to have *summable variation* if $\Sigma_{n \in \mathbf{N}} \operatorname{var}_n(g) < \infty$. Now assume that $g : X_P \to \mathbf{R}$ has summable variation. We define, for every $x \in X_P$, $m \in \mathbf{Z}$, $W^u_x(P, m)$ and $W^s_x(P, m)$ by (3.1) and (3.2) and observe that the limit

$$\mathscr{P}(g) = \lim_n \left\{ \sum_{x' \in W^u_x(P,0) \cap W^s_x(P,n)} \exp\left(\sum_{0 \leqq s < n} g(T^s_P x') \right) \right\}^{1/n} \tag{6.1}$$

exists for every $x \in X_P$ and is uniform and constant in x. The number $\mathscr{P}(g)$ appearing in (6.1) is called the *pressure* of the function g (cf. [12], [16]). Furthermore, $\Sigma_{s \in \mathbf{Z}} |g(T^s_P x) - g(T^s_P x')| < \infty$ for every $(x, x') \in R'_P$, and there exists a unique T_P-invariant probability measure μ_g on X_P which is quasi-invariant under the equivalence relation R_P and whose Radon–Nikodyn derivative r_g is given by

$$\begin{aligned} r_g(x, x') &= d\mu_g(x)/d\mu_g(x') \\ &= \exp\Big\{ \sum_{s \leqq 0} (g(T_P^{-s-m} x) - g(T_P^{-s-m'} x')) + \sum_{-m < s < n} [gT^s_P x) - \mathscr{P}(g)] \\ &\quad - \sum_{-m' < s < n'} [g(T^s_P x') - \mathscr{P}(g)] + \sum_{s \geqq 0} (g(T_P^{s+n} x) - g(T_P^{s+n'} x')) \Big\} \end{aligned} \tag{6.2}$$

for every $(x, x') \in R_P$, where m, m', n, n' are natural numbers satisfying

$$x_{-m-s} = x'_{-m'-s} \text{ and } x_{n+s} = x'_{n'+s} \tag{6.3}$$

for all $s \geqq 0$ (cf. (3.11) and (3.12), [5] and [12]). The measure μ_g is called the *Gibbs measure* (or *equilibrium state*) defined by the function $g : X_P \to \mathbf{R}$ (cf. [1], [12], [16]). We denote by G_g (F_g) the groups of all nonsingular automorphisms V of (X_P, μ_g) with the property that $Vx \in R_P(x)$ ($Vx \in R'_P(x)$) for μ_g-a.e. $x \in X_P$. Note that the group F_g may not contain any measure preserving elements other than the identity transformation $\mathbf{1}$ on X_P. We define the *information cocycle* $\mathscr{I}_g : P_P \to \mathbf{R}$ for the Gibbs measure μ_g by setting

$$\begin{aligned} \mathscr{I}_g(x, x') = &\sum_{0 \leqq s < n} [g(T^s_P x) - \mathscr{P}(g)] - \sum_{0 \leqq s < n'} [g(T^s_P x') - \mathscr{P}(g)] \\ &+ \sum_{s \geqq 0} (g(T_P^{s+n} x) - g(T_P^{s+n'} x')) \end{aligned} \tag{6.4}$$

for every $(x, x') \in R_P$ satisfying (6.3) (this definition simplifies notation and differs from the usual one by a coboundary — cf. [4] and [5]). Finally we introduce the *β-function* $\beta_g : \mathbf{R} \to \mathbf{R}_+$ given by

$$\beta_g(t) = \mathcal{P}(t \cdot [g - \mathcal{P}(g)]), \qquad t \in \mathbf{R} \qquad \text{(cf. (6.1), [4] and [5]).} \tag{6.5}$$

If $Q = (Q(j, j'), 1 \leqq j, j \leqq l)$ is a second irreducible 0–1 matrix, $h : X_Q \to \mathbf{R}$ a function with summable variation, and μ_h the corresponding Givvs measure on X_Q, consider a measure preserving isomorphism $\varphi : (X_P, \mu_g) \to (X_Q, \mu_h)$ satisfying (2.2). The isomorphism φ is said to be almost continuous, a hyperbolic structure homomorphism or a hyperbolic structure preserving isomorphism of the shifts T_P and T_Q if it satisfies (2.3)–(2.5), (3.5)–(3.6), or (3.7)–(3.8), respectively.

6.1. Theorem. *Let P and Q be irreducible 0–1 matrices, T_P and T_Q the corresponding shifts on the spaces X_P and X_Q, $g : X_P \to \mathbf{R}$ and $h : X_Q \to \mathbf{R}$ functions with summable variation, and μ_g and μ_h the associated Gibbs measures on the spaces X_P and X_Q, respectively. A measure preserving isomorphism $\varphi : (X_P, \mu_g) \to (X_Q, \mu_h)$ of T_P and T_Q is an almost continuous hyperbolic structure homomorphism if and only if there exists a null set $N_g \subset X_P$ and functions a^*_φ, $m^*_\varphi : X_P \setminus N_g \to \mathbf{N}$ such that* (4.9) *and* (4.10) *are satisfied for all $x, x' \in X_P \setminus N_g$.*

The proof of Theorem 6.1 is a minor modification of that of Theorem 4.1 and Corollary 4.2, allowing for the possible nonexistence of measure preserving transformations in F_g. For every $\delta > 0$ we can choose the automorphism V in (4.4) to satisfy that $|d\mu_g V / d\mu_g - 1| < \delta$ μ_g-a.e. If δ is sufficiently small we obtain the same contradiction which proves Theorem 4.1. The following Theorems 6.2 and 6.3 are due to A. Harding [5] in the special case of finitary isomorphisms with finite expected code lengths.

6.2. Theorem. *Let P and Q be irreducible 0–1 matrices, T_P and T_Q the corresponding shifts on the spaces X_P and X_Q, $g : X_P \to \mathbf{R}$ and $h : X_Q \to \mathbf{R}$ functions with summable variation, and μ_g and μ_h the associated Gibbs measures on the spaces X_P and X_Q, respectively. Assume that there exists a measure preserving isomorphism $\varphi : (X_P, \mu_g) \to (X_Q, \mu_h)$ of T_P and T_Q which is a hyperbolic structure homomorphism. Then one can find a measurable function $f : X_P \to \mathbf{R}$ and a null set $N \subset X_P$ such that*

$$\mathcal{I}_g(x, x') = \mathcal{I}_h(\varphi(x), \varphi(x')) + f(x) - f(x') \tag{6.6}$$

for every $(x, x') \in R_P \setminus N \times N$. In other words, the cocycles $\mathcal{I}_g$ and $\varphi_* \mathcal{I}_h$ are cohomologous (cf. [14], Remark 3.4).

The proof of Theorem 6.2 is analogous to the proof of Theorem 3.1 in [14] and will be omitted. For finitary isomorphisms with finite expected code lengths a detailed proof can be found in [5]. We remark in passing that equation (6.6) allows us to conclude that the information variance is an invariant of hyperbolic structure preserving maps (cf. [14], Remark 3.2).

6.3. THEOREM. *Let P and Q be irreducible 0–1 matrices, T_P and T_Q the corresponding shifts on the spaces X_P and X_Q, $g: X_P \to \mathbf{R}$ and $h: X_Q \to \mathbf{R}$ functions with summable variation, and μ_g and μ_h the associated Gibbs measures on the spaces X_P and X_Q, respectively. Assume that there exists an almost continuous measure preserving isomorphism $\varphi: (X_P, \mu_g) \to (X_Q, \mu_h)$ of T_P and T_Q which is a hyperbolic structure homomorphism. Then $\beta_g(t) \leqq \beta_h(t)$ for every $t \in \mathbf{R}$.*

PROOF. This is proved in exactly the same way as Theorem 5.1. We define $D(n)$, $n \geqq 0$, by (5.2) and obtain (5.3)–(5.6). Since

$$d\mu_g(x)/d\mu_g(V_n x) = \exp\Big\{ \sum_{s<0} (g(T_P^s x) - g(T_P^s V_n x)) + \sum_{0 \leqq s < n} [g(T_P^s x) - \mathcal{P}(g)] + \sum_{s \geqq n} (g(T_P^s x) - g(T_P^{s-n} V_n x)) \Big\}$$

(cf. (6.2)), it follows that

$$C(g)^{-4} \cdot \exp\Big\{ \sum_{0 \leqq s < n} [g(T_P^s x) - \mathcal{P}(g)] \Big\} \leqq d\mu_g(x)/d\mu_g(V_n x) \leqq C(g)^4 \cdot \exp\Big\{ \sum_{0 \leqq s < n} [g(T_P^s x) - \mathcal{P}(g)] \Big\} \tag{6.7}$$

and, similarly, that

$$C(h)^{-4} \cdot \exp\Big\{ \sum_{0 \leqq s < n} [h(T_Q^s \varphi(x)) - \mathcal{P}(h)] \Big\} \leqq d\mu_h(\varphi(x))/d\mu_h(\varphi(V_n x)) \leqq C(h)^4 \cdot \exp\Big\{ \sum_{0 \leqq s < n} [h(T_Q^s x) - \mathcal{P}(h)] \Big\} \tag{6.8}$$

for every $n \geqq 1$ and $x \in D(n) \setminus N$, where $C(g) = \exp\{\Sigma_{n \in \mathbf{N}} \operatorname{var}_n(g)\}$ and $C(h) = \exp\{\Sigma_{n \in \mathbf{N}} \operatorname{var}_n(h)\}$. When combining (6.5) and (6.6) with the assumption that $\mu_g \varphi^{-1} = \mu_h$ we conclude that

$$\exp\Big\{ \sum_{0 \leqq s < n} [g(T_P^s x) - \mathcal{P}(g)] \Big\} \leqq C(g)^4 \cdot C(h)^4 \cdot \exp\Big\{ \sum_{0 \leqq s < n} [h(T_Q^s \varphi(x)) - \mathcal{P}(h)] \Big\} \tag{6.9}$$

for every $n \geqq 1$ and every $x \in D(n) \setminus N$. For every $t \in \mathbf{R}$, $n \geqq 1$, $x \in D(n) \setminus N$ we thus have that

$$\left\{ \sum_{x' \in W^u_x(P,M) \cap W^s_x(P,n-M)} \exp t \cdot \left(\sum_{0 \leqq s < n} [g(T^s_P x') - \mathcal{P}(g)] \right) \right\}$$
$$\leqq C(g)^{4t} \cdot C(h)^{4t} \cdot \left\{ \sum_{y \in W^u_{\varphi(x)}(Q,0) \cap W^s_{\varphi(x)}(Q,n)} \exp t \cdot \left(\sum_{0 \leqq s < n} [h(T^s_Q y) - \mathcal{P}(h)] \right) \right\}. \tag{6.10}$$

For every $\varepsilon > 0$ there exists an integer $n' > 0$ with

$$\left| \left\{ \sum_{x' \in W_x^u(P,M) \cap W^s_x(P,n-M)} \exp t \cdot \left(\sum_{0 \leqq s < n} [g(T^s_P x') - \mathcal{P}(g)] \right) \right\}^{1/n} \right.$$
$$\left. - \left\{ \sum_{x' \in W^u_x(P,0) \cap W^s_x(P,n)} \exp t \cdot \left(\sum_{0 \leqq s < n} [g(T^s_P x') - \mathcal{P}(h)] \right) \right\}^{1/n} \right| < \varepsilon \tag{6.11}$$

for every $n \geqq n'$ and $x \in D(n)$. The proof now follows from (6.1), (6.5), (6.10) and (6.11). □

6.4. Corollary ([5]). *Let P and Q be irreducible 0–1 matrices, T_P and T_Q the corresponding shifts on the spaces X_P and X_Q, $g : X_P \to \mathbf{R}$ and $h : X_Q \to \mathbf{R}$ functions with summable variation, and μ_g and μ_h the associated Gibbs measures on the spaces X_P and X_Q, respectively. Assume that there exists a finitary, hyperbolic structure preserving isomorphism $\varphi : (X_P, \mu_g) \to (X_Q, \mu_h)$ of T_P and T_Q. Then $\beta_g(t) = \beta_h(t)$ for every $t \in \mathbf{R}$.*

6.5. Problem. Do there exist satisfactory and computable analogues of the groups Γ_P and Δ_P for Gibbs measures?

The invariants derived in this section (i.e. the cohomology class of $\mathcal{I}_g$ and the function β_g) indicate that one cannot — in general — find (finitary) hyperbolic structure preserving isomorphisms between Markov shifts T_P and T_Q which send a specified Gibbs measure μ_g on X_P to a specified Gibbs measure μ_h on X_Q, even if the entropies and periods of the shifts coincide. Nevertheless there exist some natural examples of such isomorphisms which arise in a geometric context (for notation, terminology and details we refer to [2] and [12]). Let Ω be a basic set of a C^2 Axiom A diffeomorphism f, $\chi : \Omega \to \mathbf{R}$ a Hölder continuous function, and ν_χ the unique equilibrium state of f on Ω determined by χ. Choose a Markov partition $\mathcal{P} = \{R_1, \ldots, R_k\}$ of Ω of sufficiently small diameter such that $\mathcal{P}$ is a generator (we shall call such a partition a Markov generator of f on Ω), denote by P the 0–1 matrix corresponding to $\mathcal{P}$ and write T_P for the shift on the associated shift space X_P. The obvious projection $\theta : X_P \to \Omega$ is continuous and surjective, and the function $g = \chi \cdot \theta$ on X_P has summable variation. The Gibbs measure μ_g on X_P satisfies that $\mu_g\theta^{-1} = \nu_X$, and there exists a ν_X-null set $N \subset \Omega$ such that the restriction of θ to $X_P \setminus \theta^{-1}(N)$ is a homeomorphism of $X_P \setminus \theta^{-1}(N)$ onto $\Omega \setminus N$. Since $\mathcal{P}$ is a Markov partition there exists a null set N' in Ω such that $\theta(W^s_x \setminus \theta^{-1}(N')) = W^s_{\theta(x)} \setminus N'$ and $\theta(W^u_x \setminus \theta^{-1}(N')) = W^u_{\theta(x)} \setminus N'$ for every $x \in X_P$, where W^s_ω and W^u_ω denote the stable and unstable manifolds in Ω of a point $\omega \in \Omega$. The following results are now obvious.

6.6. PROPOSITION. *Let Ω be a basic set of a C^2 Axiom A diffeomorphism f, $\chi : \Omega \to \mathbf{R}$ a Hölder continuous function, and ν_χ the unique equilibrium state of f determined by χ. Let $\mathcal{P}$ and $\mathcal{Q}$ be Markov generators of f on Ω and let T_P and T_Q be the shifts on the shift spaces X_P and X_Q corresponding to $\mathcal{P}$ and $\mathcal{Q}$. Denote by $\theta : X_P \to \Omega$ the natural projection maps, put $g = \chi \cdot \theta$, $h = \chi \cdot \eta$, and write μ_g and μ_h for the Gibbs measures on X_P and X_Q corresponding to g and h, respectively. Then there exists a measure preserving isomorphism $\varphi : (X_P, \mu_g) \to (X_Q, \mu_h)$ of the shifts T_P and T_Q which is finitary and hyperbolic structure preserving.*

6.7. THEOREM. *Consider two C^2 Axiom A diffeomorphisms f_i, restricted to basic sets Ω_i, with equilibrium states ν_i arising from Hölder continuous functions $\chi_i : \Omega_i \to \mathbf{R}$. The following conditions are equivalent.*

(1) *There exist Markov generators $\mathcal{P}_i$ of f_i on Ω_i such that the associated Markov shifts T_i on the shift spaces (X_i, μ_i) arising from $\mathcal{P}_i$, $i = 1,2$* (cf. Proposition 6.6) *are isomorphic via a finitary, hyperbolic structure preserving isomorphism.*

(2) *For all Markov generators $\mathcal{P}_i$ of f_i on Ω_i, $i = 1,2$, the associated Markov shifts T_i on the shift spaces (X_i, μ_i) arising from $\mathcal{P}_i$, $i = 1,2$, are isomorphic via finitary, hyperbolic structure preserving isomorphisms.*

(3) *There exist ν_i-null sets $N_i \subset \Omega_i$ and a homeomorphism $\psi : \Omega_i \setminus N_1 \to \Omega_2 \setminus N_2$ such that $\nu_1\psi^{-1} = \nu_2$, $\psi(f_1\omega) = f_2\psi(\omega)$ for every $\omega \in \Omega_1 \setminus N_1$, and $\psi(W^s_\omega \setminus N_1) = W^s_{\psi(\omega)} \setminus N_2$, $\psi(W^u_\omega \setminus N_1) = W^u_{\psi(\omega)} \setminus N_2$ for every $\omega \in \Omega_1$.*

6.8. EXAMPLE. *The Reversibility of Linear Hyperbolic Toral Automorphisms.* Let A be a linear hyperbolic automorphism of the n-torus $\mathbf{R}^n/\mathbf{Z}^n$ with Lebesgue measure λ. We write W^s_ω and W^u_ω for the stable and unstable manifolds of a point $\omega \in \mathbf{R}^n/\mathbf{Z}^n$ under A. Then there exists a null set $N \subset \mathbf{R}^n/\mathbf{Z}^n$ and a Lebesgue measure preserving homeomorphism $\psi : \mathbf{R}^n/\mathbf{Z}^n \setminus N \to \mathbf{R}^n/\mathbf{Z}^n \setminus N$ such that $\psi(A\omega) = A^{-1}\psi(\omega)$ for every $\omega \in \mathbf{R}^n/\mathbf{Z}^n \setminus N$, $\psi(W^s_\omega \setminus N) = W^u_{\psi(\omega)} \setminus N$, and $\psi(W^u_\omega \setminus N) = W^s_{\psi(\omega)} \setminus N$ for every $\omega \in \mathbf{R}^n/\mathbf{Z}^n$ (note that W^u_ω and W^s_ω are the stable and unstable manifolds, respectively, of A^{-1}). The existence of such a map is quite interesting, since W^s_ω and W^u_ω may have different dimensions. In order to prove the existence of ψ, choose a Markov generator $\mathcal{P}$ of Ω and consider the associated Markov shift T_P on the shift space X_P with natural projection $\eta : X_P \to \Omega$. We denote by μ the measure of maximal entropy on X_P (which is Markov) and observe that $\mu\eta^{-1} = \lambda$. The existence of the map ψ described above follows from Theorem 6.7 and from the well known fact that there exists a μ-null set $N' \subset X_P$ and a μ-preserving homeomorphism $\varphi : X_P \setminus N' \to X_P \setminus N'$ such that $\varphi \cdot T_P = T_P^{-1} \cdot \varphi$ on $T_P \setminus N'$ and $\varphi(W_x^s \setminus N') = W_x^u \setminus N'$, $\varphi(W_x^u \setminus N') = W_x^s \setminus N'$ for every $x \in X_P$, where W_x^s and W_x^u denote the stable and unstable manifolds of a point $x \in X_P$.

An example of such a map φ (due to R. Butler [3] and W. Parry — cf. [11], p. 52) can be obtained as follows: if $P = (P(i, i'), 1 \leqq i, i' \leqq k)$, $k \geqq 2$, is an arbitrary, irreducible Markov matrix and T_P the Markov shift on the shift space X_P with Markov measure m_P, put $N' = \bigcap_{m \geqq 1} (\bigcup_{n \geqq m} [1]_n \cap \bigcup_{n \geqq m} [1]_{-n})$, where $[1]_j = \{x \in X_P : x_j = 1\}$. By using the symbol "1" as a marker, every $x \in X_P \setminus N'$ can be written as

$$\ldots, i_{-2,1}, \ldots, i_{-2,j(-2)}, i_{-1,1}, \ldots, i_{-1,j(-1)}, i_{0,1}, \ldots, i_{0,j(0)}, i_{1,1}, \ldots, i_{1,j(1)}, i_{2,1}, \ldots, i_{2,j(2)}, \ldots, \tag{6.12}$$

where $1 \leqq j(n) < \infty$, $i_{n,j} \neq 1$ for $1 \leqq j < j(n)$, and $i_{n,j(n)} = 1$ for all $n \in \mathbf{Z}$, and where the zero coordinate of x lies amongst the coordinates $i_{0,1}, \ldots, i_{0,j(0)}$. For x given by (6.12) we define $\varphi(x)$ by

$$\ldots, i_{2,1}, \ldots, i_{2,j(2)}, i_{1,1}, \ldots, i_{1,j(1)}, i_{0,1}, \ldots, i_{0,j(0)}, i_{-1,1}, \ldots, i_{-1,j(-1)}, i_{-2,1}, \ldots, i_{-2,j(-2)}, \ldots .$$

If the zero coordinate of x is $i_{0,s}$, $1 \leqq s < j(0)$, then $\varphi(x)$ has the zero coordinate $i_{0,j(0)-s}$; if x has zero coordinate $i_{0,j(0)}$, the zero coordinate of $\varphi(x)$ is defined to be $i_{1,j(1)}$. The map $\varphi : X_P \setminus N' \to X_P \setminus N'$ is obviously a measure preserving homeomorphism with $\varphi(W_x^s \setminus N') = W_x^u \setminus N'$, $\varphi(W_x^u \setminus N') = W_x^s \setminus N'$ for every $x \in X_P$, and such that $\varphi \cdot T_P = T_P^{-1} \cdot \varphi$ on $X_P \setminus N'$.

6.9. Problem. Let P be an irreducible 0–1 matrix, T_P the corresponding Markov shift on the shift space X_P, $g : X_P \to \mathbf{R}$ a function with summable variation, and μ_g the corresponding Gibbs measure on X_P. Does there exist a μ_g-null set $N \subset X_P$ and a hormeomorphism $\varphi : X_P \setminus N \to X_P \setminus N$ such that $\varphi \cdot T_P = T_P \cdot \varphi$ on $X_P \setminus N$ and $\varphi(W_x^s \setminus N) = W_{\varphi(x)}^u \setminus N$, $\varphi(W_x^u \setminus N) = W_{\varphi(x)}^s \setminus N$ for every $x \in X_P$, i.e. are T_P and T_P^{-1} isomorphic via a finitary, hyperbolic structure preserving isomorphism $\varphi : (X_P, \mu_g) \to (X_P, \mu_g)$? If such an isomorphism always exists then Theorem 6.7 implies that every C^2 Axiom A diffeomorphism f, restricted to a basic set Ω, with equilibrium state ν_χ arising from Hölder continuous functions $\chi : \Omega \to \mathbf{R}$, is reversible in the sense of Example 6.8.

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